Ergodic Theory - Week 9

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1 Classifying measure preserving systems

P1. **(a)** Show that if the system $(X, \mathcal{B}, \mu, T)$ is mixing, then for all strictly increasing sequences of positive integers n_k and any $A \in \mathcal{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k} A\right) = 1.$$

(b) Show that if the system $(X, \mathcal{B}, \mu, T)$ is weak-mixing, then for all sequences of positive integers n_k with positive density and any $A \in \mathcal{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k} A\right) = 1.$$

(c) ** Show that the converse in part (b) holds as well.
Hint: Use the fact that if a system has an eigenfunction, then it has a factor map to a rotation system.

P2. **(a)** Let $(X, \mathcal{B}, \mu, T)$ be a weakly-mixing system. Show that for all $a \in (0, 1)$ and any $f \in L^\infty(X)$, we have that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n a} f(T^n x) = 0$$

for almost all $x \in X$.
Hint: See also exercise 1 in Week 4.

(b) Let $(X, \mathcal{B}, \mu, T)$ be a mixing system. Show that for all $f \in L^\infty(X)$, we have

$$\lim_{N \to +\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} f - \int f d\mu \right\|_{L^2(X)} = 0.$$

Hint: $|2^n - 2^m|$ is "large" for "almost all" pairs (n, m).

P3. A measure preserving system $(X, \mathcal{B}, \mu, T)$ is called *rigid* if there is an increasing sequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that for all $f \in L^2(X, \mathcal{B}, \mu)$, we have $\|f \circ T^{n_k} - f\|_2 \to 0$ as $k \to \infty$.

(a) Prove that $(X, \mathcal{B}, \mu, T)$ is rigid if and only if there is an increasing sequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, a dense subset $V \subseteq L^2(X, \mathcal{B}, \mu)$ such that for all $f \in V$, $\|f \circ T^{n_k} - f\|_2 \to 0$ as $k \to \infty$.

(b) Suppose that $(X, \mu, \mathcal{B}, T)$ has discrete spectrum. Prove that $(X, \mu, \mathcal{B}, T)$ is rigid.
Hint: Use part **(a)** for a suitable dense subspace $V \subseteq L^2(X, \mathcal{B}, \mu)$, and also use **P2** from the Exercise sheet 6.

(c) We call a system $(X, \mathcal{B}, \mu, T)$ mildly mixing if it has no non-trivial rigid factors. Namely, there does not exist a factor map $(X, \mathcal{B}, \mu, T) \to (Y, \mathcal{A}, \nu, S)$ such that the system $(Y, \mathcal{A}, \nu, S)$ is rigid (and non-trivial).

Show that a mixing system is mildly mixing.

P4. Consider the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), \mu, T)$ where μ is the Haar measure in $\mathbb{T}^2$ and T is the baker's map defined by

$$T(x, y) = (2x - \lfloor 2x \rfloor, \frac{y + \lfloor 2x \rfloor}{2}).$$

Prove that this map is a Bernoulli system.